数理解析研究所講究録
第 1659 巻 2009 年 141-156

Spherical functions on $U(n,n)/(U(n)\times U(n))$ and hermitian Siegel series

Yumiko Hironaka

§0 Introduction

Let k' be an unramified quadratic extension over a non-archimedean local filed k of characteristic 0. We fix a prime element π of k, and the additive value $v_\pi(\ )$ and the normalized absolute value $|\ |$ on $k^\times$, where $|\pi|^{-1}=q$ is the cardinality of the residue class field of k. We consider hermitian matrices with respect to the involution $*$ on k' which is identity on k, and set

$$\mathcal{H}_m=\{A\in M_m(k')\mid A^*=A\},\quad \mathcal{H}_m^{nd}=\mathcal{H}_m\cap GL_m(k'), \tag{0.1}$$

where, for a matrix $A=(a_{ij})\in M_{mn}(k')$, we denote by A^* the matrix $(a_{ji}{}^*)\in M_{nm}(k')$.

For $T\in\mathcal{H}_n^{nd}$, we define the spaces

$$\mathfrak{X}_T=\{x\in M_{2n,n}(k')\mid x^*H_nx=T\},\quad X_T=\mathfrak{X}_T/U(T),$$

where $H_n=\begin{pmatrix}0 & 1_n\\ 1_n & 0\end{pmatrix}\in\mathcal{H}_{2n}$ and $U(T)=\{g\in GL_n(k')\mid g^*Tg=T\}$. We consider spherical functions on X_T, which is isomorphic to $U(n,n)/(U(T)\times U(T))$ over k, where $U(n,n)=U(H_n)$ (cf. Lemma 1.1). We consider the following integral

$$\omega_T(\overline{x};s)=\int_K|f_T(kx)|^{s+\varepsilon}\,dk,\quad (\overline{x}\in X_T,\ s\in\mathbb{C}^n). \tag{0.2}$$

Here dk is the normalized Haar measure on $K=U(n,n)\cap GL_{2n}(\mathcal{O}_{k'})$,

$$\varepsilon=(-1,\ldots,-1,-\tfrac{1}{2})+(\tfrac{\pi\sqrt{-1}}{\log q},\ldots,\tfrac{\pi\sqrt{-1}}{\log q})\in\mathbb{C}^n,$$

$$|f_T(x)|^s=\prod_{i=1}^n\left|d_i(x_2T^{-1}x_2^*)\right|^{s_i},$$

where x_2 is the lower half n by n block of $x\in\mathfrak{X}_T$ and $d_i(y)$ is the determinant of the upper left i by i block of y. The right hand side of (0.2) is absolutely convergent

2000 Mathematics Subject Classification: Primary 11F85; secondly 11E95, 11F70, 22E50.
Key words and phrases: spherical functions, unitary groups, hermitian Siegel series.
This research is partially supported by Grant-in-Aid for scientific Research (C):20540029.
A full version of this article will appear elsewhere (cf. arXiv : 0904.4304).

if $\mathrm{Re}(s_i) \geq 1$ $(1 \leq i \leq n-1)$ and $\mathrm{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes a common eigen function with respect to the action of Hecke algebra $\mathcal{H}(G, K)$ with $G = U(n,n)$; thus we have a spherical function on X_T. It is convenient to introduce the new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \qquad s_n = -z_n, \tag{0.3}$$

and we write $\omega_T(\overline{x}; z) = \omega_T(\overline{x}; s)$. We denote by W the Weyl group of G with respect to the maximal k-split torus in G, which is is isomorphic to $S_n \ltimes (C_2)^n$, S_n acts on z_i by permutation of indices. We denote by Σ^+ the set of positive roots of G with respect to the Borel group, and regard it a subset of $\mathbb{Z}^n$ and write $\langle \alpha, z \rangle = \sum_{i=1}^n \alpha_i z_i$ for $\alpha \in \Sigma^+$ (for details, see §2.2).

Our main results in §1 and §2 are the following.

Theorem 1(i) *For any $T \in \mathcal{H}_n^{nd}$, the function*

$$\prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})}{(1 - q^{z_i - z_j - 1})} \times \omega_T(\overline{x}; z)$$

is holomorphic for all z in $\mathbb{C}^n$ and S_n-invariant, and the function

$$|2|^{-z_1 - z_2 - \cdots - z_n} \prod_{1 \leq i < j \leq n} \frac{(1 + q^{z_i - z_j})(1 + q^{z_i + z_j})}{(1 - q^{z_i - z_j - 1})(1 - q^{z_i + z_j - 1})} \times \omega_T(\overline{x}; z)$$

is also holomorphic for all z in $\mathbb{C}^n$ and W-invariant. In particular the latter is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^W$.

(ii) *For any $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the following functional equation holds*

$$\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \tag{0.4}$$

where

$$\Gamma_\sigma(z) = \prod_{\substack{\alpha \in \Sigma^+ \\ \sigma(\alpha) < 0}} f_\alpha(\langle \alpha, z \rangle), \qquad f_\alpha(t) = \begin{cases} \dfrac{1 - q^{t-1}}{q^t - q^{-1}} & \text{if } \alpha \text{ is short} \\ |2|^t & \text{if } \alpha \text{ is long} \end{cases}.$$

In §3, we give an explicit expression for $\omega_T(x_T; s)$.

As an application, we consider the hermitian Siegel series in §4. For each $T \in \mathcal{H}_n$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\mathrm{tr}(TR)) dR, \tag{0.5}$$

where ψ is an additive character on k of conductor $\mathcal{O}_k$, tr( ) is the trace of matrix and $\nu_\pi(R)$ is the "denominator" of R, which is certain non-negative powers of q (cf. (4.1)) As for Siegel series (for symmetric matrices), F. Sato and the author have given a new integral expression and related it to a spherical function on the symmetric space $O(2n)/(O(n) \times O(n))$ (cf. [HS]). In the present paper we develop the similar argument for hermitian Siegel series. Since we know well about the functional equations of spherical functions $\omega_T(\overline{x}; s)$ with respect to W as above, we can bring out the functional equation of $b_\pi(T; s)$ as an application; thus we obtain an integral expression of $b_\pi(T; s)$ and its functional equation.

Theorem 2(i) *If* $\mathrm{Re}(s) > 2n$, *one has*

$$b_\pi(T; s) = \zeta_n(k'; \frac{s}{2})^{-1} \cdot \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} \left|N_{k'/k}(\det x_2)\right|^{\frac{s}{2}-n} \left|\Theta_T\right|(x), \tag{0.6}$$

where $\mathfrak{X}_T(\mathcal{O}_{k'}) = \mathfrak{X}_T \cap M_{2n,n}(\mathcal{O}_{k'})$, $\zeta_n(k'; \ )$ *is the zeta function of the matrix algebra* $M_n(k')$, *and* $\left|\Theta_T\right|(x)$ *is a certain normalized measure on* $\mathfrak{X}_T$.

(ii) *For any* $T \in \mathcal{H}_n^{nd}$, *one has*

$$\frac{b_\pi(T; s)}{\prod_{i=0}^{n-1}(1 - (-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} \left|\det(T/2)\right|^{s-n} \times \frac{b_\pi(T; 2n-s)}{\prod_{i=0}^{n-1}(1 - (-1)^i q^{-(2n-s)+i})},$$

where χ_π *is the character on* $k^\times$ *determined by*

$$\chi_\pi(a) = (-1)^{v_\pi(a)} = |a|^{\frac{\pi\sqrt{-1}}{\log q}}, \quad a \in k^\times.$$

We note here that the above functional equation is related to an element of the Weyl group of $U(n, n)$, which was not the case for symmetric case when n is odd. The existence of functional equation of $b_\pi(T; s)$ was known in an abstract form as functional equations of Whitakker functions of a p-adic group by Karel [Kr](cf. also Kudla-Sweet [KS], Ikeda [Ik]).

§1

We follow the notations in the introduction. For $A \in \mathcal{H}_m$ and $X \in M_{mn}(k')$, we write

$$A[X] = X^* A X = X^* \cdot A \in \mathcal{H}_n,$$

then our spaces are given for each $T \in \mathcal{H}_n^{nd}$ by

$$\mathfrak{X}_T = \{x \in M_{2n,n}(k') \mid H_n[x] = T\}, \quad X_T = \mathfrak{X}_T / U(T), \tag{1.1}$$

$$x_T = \begin{pmatrix} \frac{1}{2}T \\ 1_n \end{pmatrix} \in \mathfrak{X}_T.$$

The group $G = U(n, n)$ acts on $\mathfrak{X}_T$, as well as on X_T, through left multiplication, which is transitive by Witt's theorem for hermitian matrices (cf. [Sch], Ch.7, §9). Our first observation is the following.

Lemma 1.1 *The stabilizer subgroup of $G = U(n,n)$ at $x_T U(T) \in X_T$ is given as*

$$\left\{ \widetilde{T}^{-1} \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \widetilde{T} \,\middle|\, h_1,\, h_2 \in U(T) \right\}, \qquad \widetilde{T} = \begin{pmatrix} 1_n & \frac{1}{2}T \\ 1_n & -\frac{1}{2}T \end{pmatrix} \in GL_{2n}(k').$$

In particular, the space X_T is isomorphic to $G/\left(U(T) \times U(T)\right)$.

We fix the Borel subgroup B of G as

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \,\middle|\, \begin{array}{l} b \text{ is upper triangular of size } n, \\ a + a^* = 0 \end{array} \right\}, \tag{1.2}$$

and introduce the B-relative invariants on $\mathfrak{X}_T$

$$f_{T,i}(x) = d_i(x_2 T^{-1} x_2^*) \quad 1 \le i \le n, \tag{1.3}$$

associated with k-rational characters ψ_i of B by

$$f_{T,i}(bx) = \psi_i(b) f_{T,i}(x), \quad \psi_i(b) = N(d_i(b))^{-1}, \tag{1.4}$$

where x_2 is the lower half n by n block of $x \in \mathfrak{X}_T$, $d_i(y)$ is the determinant of upper left i by i block of y and $N = N_{k'/k}$. Since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$, we understand $f_{T,i}(x)$ as B-relative invariants on X_T, $1 \le i \le n$.

Remark 1.2 It is possible to realize above objects as the sets of k-rational points of algebraic sets defined over k and develop the arguments, but we take down to earth way for simplicity of notations. We only note here that X_T is isomorphic to $U(n,n)/(U(n) \times U(n))$ over the algebraic closure $\overline{k}$ of k and $\{x \in X_T \mid f_{T,i}(x) \ne 0,\ 1 \le i \le n\}$ is a Zariski open B-orbit over $\overline{k}$, where $U(n) = U(1_n)$.

Hereafter, we write an element $\overline{x} = xU(T)$ in X_T by its representative x in $\mathfrak{X}_T$ for simplicity of notations. We set $|0| = 0$ for the absolute value on $k^\times$ for convenience.

The modulus character δ on B (which is characterized by $d_l(bb') = \delta(b')^{-1} d_l(b)$ for the left invariant measure $d_l(b)$ on B) is given by

$$\delta^{\frac{1}{2}}(b) = \prod_{i=1}^{n-1} |\psi_i(b)|^{-1} \times |\psi_n(b)|^{-\frac{1}{2}}.$$

Now we introduce the spherical function $\omega(x;s)$ on $X_T = \mathfrak{X}_T/U(T)$

$$\omega_T(x;s) = \omega_T^{(n)}(x;s) = \int_K |f_T(kx)|^{s+\varepsilon}\, dk, \tag{1.5}$$

where dk is the normalized Haar measure on $K = G \cap GL_{2n}(\mathcal{O}_{k'})$, $s \in \mathbb{C}^n$

$$\varepsilon = (-1, \dots, -1, -\tfrac{1}{2}) + \left(\tfrac{\pi\sqrt{-1}}{\log q}, \dots, \tfrac{\pi\sqrt{-1}}{\log q}\right) \in \mathbb{C}^n,$$

$$f_T(x) = \prod_{i=1}^n f_{T,i}(x), \qquad |f_T(x)|^s = \prod_{i=1}^n |f_{T,i}(x)|^{s_i}.$$

The right hand side of (1.5) is absolutely convergent if $\mathrm{Re}(s_i) \geq 1$ $(1 \leq i \leq n-1)$ and $\mathrm{Re}(s_n) \geq \frac{1}{2}$, continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes a common eigenfunction with respect to the action of the Hecke algebra $\mathcal{H}(G,K)$ (cf. [H2], §1).

Since we see

$$\omega_{T[h]}(x;s) = \omega_T(xh^{-1};s), \qquad h \in GL_n(k'),\ x \in \mathfrak{X}_{T[h]}, \tag{1.6}$$

it suffices to consider only for diagonal T's for the study of functional properties of $\omega_T(x;s)$ (e.g., Theorem 1 in the introduction),

We write $\omega_T(x;z) = \omega_T(x;s)$ for the new variable z introduced by (1.7). The Weyl group W of G relative to the maximal k-split torus in B acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1}bn_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega_T(x;s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices and W is generated by S_n and $\tau : (z_1, \ldots, z_n) \longmapsto (z_1, \ldots, z_{n-1}, -z_n)$.

By using a result on spherical functions on the space of hermitian forms((cf. [H1]-§2 or [H3]-§4.2)), we obtain the following.

Theorem 1.3 *For any $T \in \mathcal{H}_n^{nd}$, the function*

$$\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i - 1}} \times \omega_T(x;z)$$

is holomorphic for any z in $\mathbb{C}^n$ and S_n-invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \ldots, q^{\pm z_n}]^{S_n}$.

Remark 1.4 For the transposition $\tau_i = (i\ i+1) \in W$, $1 \leq i \leq n-1$, the following functional equation holds by Theorem 1.3

$$\omega_T(x;z) = \frac{1 - q^{z_i - z_{i+1} - 1}}{q^{z_i - z_{i+1}} - q^{-1}} \times \omega_T(x;\tau_i(z)), \quad 1 \leq i \leq n-1. \tag{1.7}$$

On the other hand, one can obtain (1.8) directly in the similar way to the case of τ in § 3, then Theorem 1.3 follows from (1.8).

§2

2.1. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$ and $\varepsilon \in 1 + 4\mathcal{O}_k^\times$ if k is dyadic (cf. [Om]-63.3 and 63.4).

Theorem 2.1 *For any $T \in \mathcal{H}_n^{nd}$, the spherical function satisfies the following functional equation:*

$$\omega_T(x;z) = |2|^{2z_n}\, \omega_T(x;\tau(z)).$$

The case $n = 1$ is easy; we calculate spherical functions explicitly for representatives of K_1-orbits in $\mathfrak{X}_T$, where $K_1 = U(H_1) \cap GL_2(\mathcal{O}_{k'})$, and obtain the functional equation. For $n \geq 2$ we take a representative w_τ of $\tau \in W$ by

$$w_\tau = \left(\begin{array}{cc|cc} 1_{n-1} & & & \\ & 0 & & 1 \\ \hline & & 1_{n-1} & \\ & 1 & & 0 \end{array}\right) \in G,$$

and take the parabolic subgroup $P = P_\tau$ attached to τ (cf. [Bo], 21.11)

$$\begin{aligned} P &= B \cup Bw_\tau B \\ &= \left\{ \left(\begin{array}{cc|cc} q & & & \\ & a & & b \\ \hline & & q^{*-1} & \\ & c & & d \end{array}\right) \left(\begin{array}{cc|cc} 1_{n-1} & \alpha & & \\ & 1 & & \\ \hline & & 1_{n-1} & \\ & & -\alpha^* & 1 \end{array}\right) \left(\begin{array}{c|cc} 1_n & B & \beta \\ & -\beta^* & 0 \\ \hline & 1_n & \end{array}\right) \in G \;\middle|\; \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1),\ \alpha, \beta \in M_{n-1,1}(k'), \\ B \in M_{n-1}(k'),\ B + B^* = 0 \end{array} \right\}, \end{aligned} \tag{2.1}$$

where each empty place in the above expression means zero-entry. Hereafter we fix a diagonal $T \in \mathcal{H}_n^{nd}$, and write $f_i(x) = f_{T,i}(x)$ by abbreviating the suffix T. The B-relative invariants $f_i(x)$ become P-relative invariants associated with ψ_i except $i = n$. We consider the following action of $\widetilde{P} = P \times GL_1$ on $\widetilde{X}_T = X_T \times V$ with $V = M_{21}(k')$:

$$(p, r) \cdot (x, v) = (px, \rho(p) v r^{-1}),$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the decomposition of $p \in P$ as in (2.1). For $(x, v) \in \widetilde{X}_T$, set

$$g(x, v) \;=\; \det \left[\left(\begin{array}{c|c} 1_{n-1} & \\ \hline & {}^t v \end{array}\right) \begin{pmatrix} x_2 \\ -y \end{pmatrix} \cdot T^{-1} \right],$$

where x_2 is the lower half n by n block of x (the same before) and y is the n-th row of x.

Then we obtain

Lemma 2.2 *$g(x, v)$ is a relative $\widetilde{P}$-invariant on $\widetilde{X_T}$ associated with character*

$$\widetilde{\psi}(p, r) = N(d_{n-1}(p))^{-1} N(r)^{-1} = \psi_{n-1}(p) N(r)^{-1}, \quad (p, r) \in \widetilde{P} = P \times GL_1,$$

satisfies

$$g(x, v_0) = f_n(x), \qquad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and is expressed as

$$g(x, v) = D(x)[v],$$

with some hermitian matrix

$$D(x) = \begin{pmatrix} a & b + c\sqrt{\epsilon} \\ b - c\sqrt{\epsilon} & d \end{pmatrix}, \quad a, c, d \in k,\ b = -\frac{1}{2} f_{n-1}(x),\ ad = b^2 - c^2\epsilon. \tag{2.2}$$

In order to prove Theorem 2.2, we need the functional equation of the following function

$$\zeta_{K_1}(A;s) = \int_{K_1} |d_1(h\cdot A)|^{s-\frac{1}{2}}\, dh, \qquad (A \in \mathcal{H}_2,\ s \in \mathbb{C}),$$

where dh is the normalized Haar measure on K_1.

Lemma 2.3 *Let $x \in X_T$ such that $f_T(x) \neq 0$ and $D(x)$ be given by (2.4). Then one has*

$$\zeta_{K_1}(D(x),s) = |2|^{-2s}\, |f_{n-1}(x)|^{2s}\, \zeta_{K_1}(D(x),-s).$$

Now Theorem 2.2 is proved as follows. By the embedding

$$K_1 \longrightarrow K = K_n, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \widetilde{h} = \left(\begin{array}{cc|cc} 1_{n-1} & & & \\ & a & & b \\ \hline & & 1_{n-1} & \\ & c & & d \end{array}\right),$$

we have

$$\begin{aligned} \omega_T(x;s) &= \int_{K_1} dh \int_K \left|f(\widetilde{h}kx)\right|^{s+\varepsilon} dk \\ &= \int_K \chi_\pi(\prod_{i<n} f_i(kx)) \prod_{i<n} |f_i(kx)|^{s_i-1} \left(\int_{K_1} \chi_\pi(f_n(\widetilde{h}kx))\left|f_n(\widetilde{h}kx)\right|^{s_n-\frac{1}{2}} dh\right) dk. \end{aligned}$$

By definition of $f_n(x)$ and $g(x,v)$ and Lemma 2.3, we see

$$f_n(\widetilde{h}x) = g(x, \begin{pmatrix} d \\ -c \end{pmatrix}) = D(x)\left[\begin{pmatrix} d \\ -c \end{pmatrix}\right] = d_1(h^{*-1}\cdot D(x)), \quad (h \in K_1),$$

hence we have

$$\omega_T(x;s) = \int_K \chi_\pi(\prod_{i<n} f_i(kx)) \prod_{i<n} |f_i(kx)|^{s_i-1}\, \zeta_{K_1}(D(kx); s_n + \tfrac{\pi\sqrt{-1}}{\log q})dk.$$

Then the functional equation of $\omega_T(x;s)$ follows from Lemma 2.4. ∎

2.2. We denote by Σ the set of roots of G with respect to the k-split torus of G contained in B and by Σ^+ the set of positive roots with respect to B. We may understand

$$\Sigma^+ = \{\, e_i - e_j,\ e_i + e_j \mid 1 \le i < j \le n\} \cup \{\, 2e_i \mid 1 \le i \le n\},$$

where $e_i \in \mathbb{Z}^n$ whose j-th component is given by the Kronecker delta δ_{ij}, and the set

$$\Sigma_0 = \{\, e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{2e_n\}$$

forms the set of simple roots. We denote by Δ the subset of W consisting of the reflections associated to elements in Σ_0. Then $\Delta = \{\tau_i \mid 1 \le i \le n-1\} \cup \{\tau\}$ generates W. We write $\alpha < 0$ if $\alpha \in \Sigma$ is negative. We see the pairing $\langle\, ,\, \rangle$ on $\Sigma \times \mathbb{C}^n$ given by

$$\langle \alpha, z\rangle = \sum_{i=1}^n \alpha_i z_i, \qquad (\alpha \in \Sigma,\ z \in \mathbb{C}^n).$$

is W-invariant. Then we obtain

Theorem 2.4 *For $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the spherical function $\omega_T(x;z)$ satisfies the following functional equation*

$$\omega_T(x;z) = \Gamma_\sigma(z) \cdot \omega_T(x;\sigma(z)), \tag{2.3}$$

where

$$\Gamma_\sigma(z) = \prod_{\substack{\alpha\in\Sigma^+ \\ \sigma(\alpha)<0}} f_\alpha(\langle \alpha, z\rangle),$$

$$f_\alpha(t) = \begin{cases} |2|^t & \text{if} \quad \alpha = 2e_i \quad \text{for some} \quad i \\ \dfrac{1-q^{t-1}}{q^t - q^{-1}} & \text{otherwise,} \end{cases}$$

in particular, the Gamma factor $\Gamma_\sigma(z)$ does not depend on T nor x.

Proof. For an element of Δ, we know the Gamma factor by (1.8) and Theorem 2.2. In general, assume that $\sigma \in W$ has the shortest expression

$$\sigma = \sigma_\ell \cdots \sigma_1,$$

with $\sigma_i \in \Delta$ associted by some $\alpha_i \in \Sigma_0$. Since the Gamma factors satisfy cocycle relations and $\langle\, ,\, \rangle$ is W-invariant, we have

$$\begin{aligned} \Gamma_\sigma(z) &= \Gamma_{\sigma_\ell}(\sigma_{\ell-1}\cdots\sigma_1(z))\cdots\Gamma_{\sigma_2}(\sigma_1(z))\cdot\Gamma_{\sigma_1}(z) \\ &= f_{\alpha_\ell}(\langle\alpha_\ell, \sigma_{\ell-1}\cdots\sigma_1(z)\rangle)\cdots f_{\alpha_2}(\langle\alpha_2,\sigma_1(z)\rangle)\cdot f_{\alpha_1}(\langle\alpha_1, z\rangle) \\ &= f_{\alpha_\ell}(\langle\sigma_1\cdots\sigma_{\ell-1}(\alpha_\ell), z\rangle)\cdots f_{\alpha_2}(\langle\sigma_1(\alpha_2), z\rangle) f_{\alpha_1}(\langle\alpha_1, z\rangle). \end{aligned}$$

Hence $\Gamma_\sigma(z)$ has the required form, since we have

$$\left\{\alpha \in \Sigma^+ \mid \sigma(\alpha) < 0\right\} = \left\{\sigma_1\cdots\sigma_{k-1}(\alpha_k) \mid 1 \le k \le \ell\right\}.$$

∎

Corollary 2.5 *Set $\rho \in W$ by*

$$\rho(z_1, \ldots, z_n) = (-z_n, -z_{n-1}, \ldots, -z_1). \tag{2.4}$$

Then

$$\Gamma_\rho(z) = |2|^{2(z_1+\cdots+z_n)} \prod_{1\le i<j\le n} \frac{1-q^{z_i+z_j-1}}{q^{z_i+z_j} - q^{-1}}. \tag{2.5}$$

Remark 2.6 The above ρ gives the functional equation of the hermitian Siegel series (cf. §4), and it is interesting that such ρ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type (C_n), which was pointed out by Y. Komori.

By Theorem 1.3 and Theorem 2.5, we obtain the following.

Theorem 2.7 *Set*

$$F(z) = \prod_{\alpha \in \Sigma^+} g_\alpha(z),$$

where, for $\alpha \in \Sigma$,

$$g_\alpha(z) = \begin{cases} |2|^{-\frac{\langle \alpha, z\rangle}{2}} & \text{if} \quad \alpha = \pm 2e_i \quad \text{for some} \quad i \\ \dfrac{1+q^{\langle \alpha, z\rangle}}{1-q^{\langle \alpha, z\rangle - 1}} & \text{otherwise} \end{cases}.$$

Then, for any $T \in \mathcal{H}_n^{nd}$, *the function* $F(z)\omega_T(x;z)$ *is holomorphic for all* z *in* $\mathbb{C}^n$ *and* W*-invariant. In particular it is an element in* $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$.

Proof. Take any $\sigma \in \Delta$ associated by $\alpha \in \Sigma_0$. Then $F(z)\omega_T(x;z)$ is σ-invariant, since $g_\alpha(\sigma z) = g_{\sigma\alpha}(z) = g_{-\alpha}(z)$ and $\Gamma_\sigma(z) = g_{-\alpha}(z)/g_\alpha(z)$. Thus, $F(z)\omega_T(x,z)$ is W-invariant, since Δ generates W. Set

$$F_1(z) = \prod_{1 \le i < j \le n} \frac{1+q^{z_i - z_j}}{1-q^{z_i - z_j - 1}}, \qquad F_2(z) = |2|^{-z_1 - \cdots - z_n} \prod_{1 \le i < j \le n} \frac{1+q^{z_i + z_j}}{1-q^{z_i+z_j-1}}.$$

Then $F(z) = F_1(z)F_2(z)$ and $F_1(z)\omega_T(x;z)$ is holomorphic in $z \in \mathbb{C}^n$ and S_n-invariant by Theorem 1.3. Hence $F(z)\omega_T(x;z)$ is holomorphic in $z \in \mathbb{C}^n$, since it is W-invariant and holomorphic for certain region e.g., $\{z \in \mathbb{C}^n \mid \mathrm{Re}(z_i) \le 0\}$. ∎

§3

3.1. In this section we give an explicit formula of $\omega_T(x;s)$ at x_T by using the general formula of Proposition 1.9 in [H2] (or Theorem 2.6 in [H4]). In order to apply it, we have to check several conditions ((A1) – (A4) in [H4]-§1), and it is obvious our (B, X_T) satisfies them except (A3), which is the same as (C) below.

Proposition 3.1 *The following condition* (C) *is satisfied.*
(C) : *For* $y \in X_T$ *such that* $f_T(y) = 0$, *there exists a character* $\psi \in \langle \psi_i \mid 1 \le i \le n\rangle$ *whose restriction to the identity component of the stabilizer of* B *at* y *is not trivial.*

Theorem 3.2 *Let* $T = Diag(\pi^{\lambda_1}, \dots, \pi^{\lambda_n})$ *with* $\lambda_1 \ge \lambda_2 \cdots \ge \lambda_n \ge v_\pi(2)$. *Then*

$$\omega_T(x_T; z) = \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i (n-i+\frac{1}{2})}(1-q^{-2})^n}{\prod_{i=1}^{2n}(1-(-1)q^{-i})} \sum_{\sigma \in W} \gamma(\sigma(z))\Gamma_\sigma(z) q^{<\lambda, \sigma(z)>}, \quad (3.1)$$

where $<\lambda, z> = \sum_{i=1}^n \lambda_i z_i$, $\Gamma_\sigma(z)$ *is defined in Theorem 2.5, and*

$$\gamma(z) = \prod_{1 \le i < j \le n} \frac{(1-q^{2z_i - 2z_j - 2})(1-q^{2z_i+2z_j-2})}{(1-q^{2z_i-2z_j})(1-q^{2z_i+2z_j})} \cdot \prod_{i=1}^n \frac{1-q^{2z_i-1}}{1-q^{2z_i}}.$$

We admit Proposition 3.1 for the moment and prove Theorem 3.2.

The set $X_T^{op} = \{x \in X_T \mid f_T(x) \neq 0\}$ becomes a disjoint union of B-orbits as follows.

$$X_T^{op} = \bigsqcup_{u \in \mathcal{U}} X_{T,u}, \quad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1},$$
$$X_{T,u} = \{x \in X_T \mid v_\pi(f_{T,i}(x)) \equiv u_1 + \cdots + u_i \pmod 2, \quad 1 \leq i \leq n-1\}.$$

We set

$$\omega_{T,u}(x;s) = \int_K |f_T(kx)|_u^{s+\varepsilon}\, dk,$$

where

$$|f_T(y)|_u^{s+\varepsilon} = \begin{cases} |f_T(y)|^{s+\varepsilon} & \text{if } y \in X_{T,u}, \\ 0 & \text{otherwise .} \end{cases}$$

For a character $\chi = (\chi_1, \dots, \chi_{n-1})$ of $\mathcal{U}$, we set

$$L_T(x;\chi;z) = \int_K \chi(f_T(kx))\, |f_T(kx)|^{s+\varepsilon}\, dk = \sum_{u \in \mathcal{U}} \chi(u)\omega_{T,u}(x;z),$$

where $\chi(u) = \prod_{i=1}^{n-1} \chi_i(u_1 + \cdots + u_i)$. Adjusting z according to χ, by adding $\frac{\pi\sqrt{-1}}{\log q}$ to z_i if necessary, we may write

$$L_T(x;\chi;z) = \omega_T(x;z_\chi).$$

Then, by the functional equations of $\omega_T(x;z)$ (Theorem 2.5), we have

$$L_T(x;\chi;z) = \Gamma_\sigma(z_\chi) L_T(x;\sigma(\chi);\sigma(z)), \qquad \sigma \in W \tag{3.2}$$

by taking suitable character $\sigma(\chi)$ of $\mathcal{U}$. If χ is the trivial character $\mathbf{1}$, then (3.2) coincides with the original functional equation of $\omega_T(x;z)$. We obtain

$$\left(\omega_{T,u}(x_T;z)\right)_u = \left(A^{-1} \cdot G(\sigma,z) \cdot \sigma A\right) \left(\omega_{T,u}(x_T;\sigma(z))\right)_u,$$

where

$$A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u} \in GL_{2^n}(\mathbb{Z}),$$

and $G(\sigma,z)$ is the diagonal matrix of size 2^n whose (χ,χ)-component is $\Gamma_\sigma(z_\chi)$. For T given as in Theorem 3.2, we obtain

$$\begin{aligned} \int_U |f_T(ux_T)|^{s+\varepsilon}\, du &= |f_T(x_T)|^{s+\varepsilon} \\ &= (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{<\lambda,z>}, \end{aligned}$$

where U is the Iwahori subgroup of K compatible with B and du is the normalized Haar measure on U. Setting

$$\delta_u(x_T,z) = \begin{cases} (-1)^{\sum_i \lambda_i(n-i+1)} q^{\sum_i \lambda_i(n-i+\frac{1}{2})} q^{<\lambda,z>} & \text{if } x_T U(T) \in X_{T,u} \\ 0 & \text{otherwise,} \end{cases}$$

we have, by Proposition 1.9 in [H2] (or its generalization Theorem 2.6 in [H4]),

$$(\omega_{T,u}(x_T;z))_u = \frac{1}{Q}\sum_{\sigma\in W}\gamma(\sigma(z))\left(A^{-1}\cdot G(\sigma,z)\cdot\sigma A\right)(\delta_u(x_T,\sigma(z)))_u,$$

where

$$Q = \sum_{\sigma\in W}[U\sigma U:U]^{-1} = \prod_{i=1}^{2n}\left(1-(-1)^i q^{-i}\right)\Big/(1-q^{-2})^n.$$

Hence we obtain

$$\begin{aligned}\omega_T(x_T;z) &= \sum_{u\in\mathcal{U}}\mathbf{1}(u)\omega_u(x_T;z)\\ &= \frac{(-1)^{\sum_i\lambda_i(n-i+1)}q^{\sum_i\lambda_i(n-i+\frac{1}{2})}}{Q}\sum_{\sigma\in W}\gamma(\sigma(z))\Gamma_\sigma(z)q^{<\lambda,\sigma(z)>}.\end{aligned}$$

∎

3.2. In order to prove Proposition 3.1, we consider the action of $G\times U(T)$ on $\mathfrak{X}_T$ by $(g,h)\circ x = gxh^{-1}$. Then, the stabilizer B_y of B at $yU(T)\in X_T$ coincides with the image $B_{(y)}$ of the projection to B of the stabilizer $(B\times U(T))_y$ at $y\in\mathfrak{X}_T$ to B. Hence the condition (C) is equivalent to the following:

(C') : For $y\in\mathfrak{X}_T$ such that $f_T(y)=0$ there exists $\psi\in\langle\psi_i \mid 1\le i\le n\rangle$ whose restriction to the identity component of $B_{(y)}$ is not trivial.

It is sufficient to prove the condition (C) (equivalently, (C')) over the algebraic closure $\overline{k}$, since, for a connected linear algebraic group $\mathbb{H}$, $\mathbb{H}(k)$ is dense in $\mathbb{H}(\overline{k})$. In the rest of this section, we consider algebraic sets over $\overline{k}$, extend the involution $*$ on k' to $\overline{k}$ and denote it by $^-$, and write $\overline{x}=(\overline{x_{ij}})$ for any matrix $x=(x_{ij})$. Since $\mathfrak{X}_T$ is isomorphic to $\mathfrak{X}_{T[h]}$ by $x\longmapsto xh$ and $B_{(x)}=B_{(xh)}$ for $h\in GL_n$, we may assume that $T=1_n$. Then, our situation is the following:

$$\begin{aligned}&\mathfrak{X}=\mathfrak{X}_{1_n}=\{x\in M_{2n,n}\mid H_n[x]=1_n\},\\ &(U(n,n)\times U(n))\times\mathfrak{X}\longrightarrow\mathfrak{X},\quad ((g,h),x)\longmapsto(g,h)\circ x = gxh^{-1}.\end{aligned}$$

We consider the set

$$\widetilde{\mathfrak{X}} = \left\{(x,y)\in M_{2n,n}\oplus M_{2n,n}\;\middle|\; {}^t yH_n x = 1_n\right\}$$

together with $GL_{2n}\times GL_n$-action defined by

$$(g,h)\star(x,y) = (gxh^{-1},\dot{g}y^t h),\quad \dot{g}=H_n{}^t g^{-1}H_n, \tag{3.3}$$

and take the Borel subgroup P of GL_{2n} by

$$P=\left\{\begin{pmatrix}p_1 & r\\ 0 & p_2\end{pmatrix}\in GL_{2n}\;\middle|\; p_1,{}^t p_2\in B_n,\ r\in M_n\right\},$$

where B_n is the Borel subgroup of GL_n consisting of upper triangular matrices.

Then, the embedding $\iota : \mathfrak{X} \longmapsto \widetilde{\mathfrak{X}}$, $x \longmapsto (x, \overline{x})$ is compatible with the actions, i.e., we have the commutative diagram

$$\begin{array}{ccccc} (U(n,n) \times U(n)) & \times & \mathfrak{X} & \xrightarrow{\circ} & \mathfrak{X} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ (GL_{2n} \times GL_n) & \times & \widetilde{\mathfrak{X}} & \xrightarrow{\star} & \widetilde{\mathfrak{X}}. \end{array}$$

For $(x,y) \in \widetilde{\mathfrak{X}}$ and $p \in P$, set

$$\widetilde{f}_i(x,y) = d_i(x_2{}^t y_2), \quad \widetilde{\psi}_i(p) = \prod_{1 \le j \le i} p_j^{-1} p_{n+j}, \quad (1 \le i \le n),$$

where x_2 (resp. y_2) is the lower half n by n block of x (resp. y), and p_j is the j-th diagonal entry of p. Then for each i, we see

$$\begin{aligned} &\widetilde{f}_i((p,r) \star (x,y)) = \widetilde{\psi}_i(p)\widetilde{f}_i(x,y), \qquad (p,r) \in P \times GL_n, \\ &\widetilde{f}_i(x,\overline{x}) = f_i(x), \quad (x \in \mathfrak{X}), \qquad \widetilde{\psi}_i|_B = \psi_i. \end{aligned}$$

We set

$$\mathcal{S} = \left\{ (x,y) \in \widetilde{\mathfrak{X}} \;\middle|\; \prod_{i=1}^{n} \widetilde{f}_i(x,y) = 0, \quad (P \times GL_n) \star (x,y) \cap \mathfrak{X} \neq \emptyset \right\}.$$

For $\alpha = (x,y) \in \widetilde{\mathfrak{X}}$, we denote by H_α the stabilizer of $P \times GL_n$ at α, and by P_α its image of the projection to P. In order to prove the condition (C), it is sufficient to show the following:

$(\widetilde{C})$: For each $\alpha \in \mathcal{S}$, there exists some $\psi \in \langle \widetilde{\psi}_i \mid 1 \le i \le n \rangle$ whose restriction to the identity component of P_α is not trivial.

We show the condition $(\widetilde{C})$ by taking suitable representatives by $P \times GL_n$-action.

(i) Assume $\alpha = (x,y) \in \mathcal{S}$ satisfies $\det(x_2) \neq 0$. Then, in the $P \times GL_n$-orbit containing α, there is $\beta = (\begin{pmatrix} 0 \\ 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ h \end{pmatrix})$ with some hermitian matrix h, further we may assume

$$h = 1_r \perp \langle 0 \rangle \perp h_1 \quad \text{or} \quad h = 1_r \perp h_2,$$

where $0 \le r \le n-1$, and for h_2, there is some i, $(1 < i \le n-r)$ such that each entry in the first row and column or in the i-th row and column is 0 except at $(1,i)$ or $(i,1)$ which are 1.

Then H_β contains the following elements, according to the above type of h,

$$(\left(\begin{array}{c|c} \delta_{r+1}(a) & \\ \hline & 1_n \end{array}\right), 1_n) \quad \text{or} \quad (\left(\begin{array}{c|c} \delta_{r+1}(a) & \\ \hline & \delta_{r+i}(a) \end{array}\right), \delta_{r+i}(a)),$$

where $\delta_j(a)$ is the diagonal matrix in GL_n whose diagonal entries are 1 except the j-th which is $a \in GL_1$. Hence we see $\widetilde{\psi}_{r+1} \not\equiv 1$ on the identity component of P_β.

(ii) The case $\alpha = (x,y) \in \mathcal{S}$ with $\det(y_2) \neq 0$ is reduced to the case $\det(x_2) \neq 0$, since $\beta = (y,x) \in \mathcal{S}$ and $H_\beta = \{ (\dot{p}, {}^t r^{-1}) \mid (p,r) \in H_\alpha \}$ and $\widetilde{\psi}_i(\dot{p}) = \widetilde{\psi}_i(p)^{-1}$.

(iii) Assume $\alpha = (x', y') \in \mathcal{S}$ satisfies $\det x'_2 = \det y'_2 = 0$. Then, in the $P \times GL_n$-orbit containing α, there is some $\beta = (x, y)$ of the following type: for some integers r_i, e_j satisfying

$$\begin{aligned} &1 \leq r_1 < r_2 < \cdots r_\ell \leq n \quad (1 \leq \ell < n), \\ &1 \leq e_1 < e_2 < \cdots < e_k \leq n \quad (k = n - \ell), \end{aligned}$$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $x_i, y_i \in M_n$ is given by

x_1 : 1 at $(r_i, k+i)$-entry for $1 \leq i \leq \ell$ and 0 at any other entry;

x_2 : 1 at (e_i, i)-entry for $1 \leq i \leq k$ and 0 at any other entry;

y_1 : the e_i-th row is the same as in x_2 for $1 \leq i \leq k$, and the j-the column is 0 if $j > k$;

y_2 : the r_i-th row is the same as in x_1 for $1 \leq i \leq \ell$, and for each i, any (i, j)-entry is 0 for $j > k$ if some (i, j')-entry is non-zero entry with $j' \leq k$.

Let $D(a)$ be the diagonal matrix in GL_n whose i-th diagonal entry is $a \in GL_1$ (resp. 1) if every (i, j)-entry of y_2 is 0 for $j \leq k$ (resp. otherwise), where the r_i-th diagonal entry of $D(a)$ is a by this choice. Then H_β contains

$$\left(\left(\begin{array}{c|c} D(a) & \\ \hline & 1_n \end{array} \right), \left(\begin{array}{c|c} 1_k & \\ \hline & a1_\ell \end{array} \right) \right),$$

and $\widetilde{\psi}_{r_i} \not\equiv 1$ on the identity component of P_β, $1 \leq i \leq \ell$. ∎

§4

We recall the hermitian Siegel series, and give its integral representation and functional equation. Let ψ be an additive character of k of conductor $\mathcal{O}_k$. For $T \in \mathcal{H}_n(k')$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; s) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-s} \psi(\mathrm{tr}(TR)) dR, \tag{4.1}$$

where tr( ) is the trace of matrix and $\nu_\pi(R)$ is defined as follows: if the elementary divisors of R with negative π-powers are $\pi^{-e_1}, \ldots, \pi^{-e_r}$, then $\nu_\pi(R) = q^{e_1 + \cdots + e_r}$, and $\nu_\pi(R) = 1$ otherwise (cf. [Sh]-§13).

In the following we assume that T is nondegenerate, since the properties of $b_\pi(T; s)$ can be reduced to the nondegenerate case. We recall the set $\mathfrak{X}_T$ for $T \in \mathcal{H}_n^{nd}(k')$

$$\mathfrak{X}_T = \mathfrak{X}_T(k') = \{ x \in M_{2n,n}(k') \mid H_n[x] = T \},$$

which is the fibre space $g^{-1}(T)$ for the polynomial map $g : M_{2n,n}(k') \longrightarrow \mathcal{H}_n(k')$, $g(x) = H_n[x]$ defined over k. We may take the measure $|\Theta_T|$ on $\mathfrak{X}_T$ induced by a k-rational differential form ω on $M_{2n,n}(k')$ satisfying $\omega \wedge g^*(dT) = dx$ where dT is the canonical gauge form on $\mathcal{H}_n(k')$, dx is the canonical gauge form on $M_{2n,n}(k')$. Then the following identity holds (cf. [Ym], [HS]-§2):

$$\begin{aligned}&\int_{\mathfrak{X}_T(k')} \phi(x)\,|\Theta_T|\,(x)\\ &= \lim_{e\longrightarrow\infty} \int_{\mathcal{H}_n(\pi^{-e})} \psi(-\mathrm{tr}(Ty)) \int_{M_{2n,n}(k')} \phi(x)\psi(\mathrm{tr}(H_n[x]y))dxdy,\end{aligned}$$

where $\phi \in \mathcal{S}(M_{2n,n}(k'))$, a locally constant compactly supported function on $M_{2n,n}(k')$ and $\mathcal{H}_n(\pi^{-e}) = \mathcal{H}_n(k') \cap M_n(\pi^{-e}\mathcal{O}_{k'})$.

The following lemma can be proved in the similar line to the case of symmetric matrices (cf. [HS]-§2).

Lemma 4.1 *If* $\mathrm{Re}(s) > n$, *one has*

$$\begin{aligned}&\int_{\mathfrak{X}_T(\mathcal{O}_{k'})} \left|N_{k'/k}(\det x_2)\right|^{s-n} |\Theta_T|\,(x) \qquad (4.2)\\ &= \lim_{e\longrightarrow\infty} \int_{\mathcal{H}_n(\pi^{-e}\mathcal{O}_{k'})} \psi(-\mathrm{tr}(Ty))dy \int_{M_{2n,n}(\mathcal{O}_{k'})} \left|N_{k'/k}(\det x_2)\right|^{s-n} \psi(\mathrm{tr}(H_n[x]y))dx.\end{aligned}$$

Let us recall the zeta function of the matrix algebra $M_n(k')$ and its explicit formula:

$$\begin{aligned}\zeta(k';s) &= \int_{M_n(\mathcal{O}_{k'})} |\det x|_{k'}^{s-n}\, dx = \int_{M_n(\mathcal{O}_{k'})} \left|N_{k'/k}(\det x)\right|^{s-n} dx\\ &= \prod_{i=1}^{n} \frac{1-q^{-2i}}{1-q^{-2(s-i+1)}}.\end{aligned}$$

Then we obtain the following integral expression of hermitian Siegel series, which can be proved in a similar line to the case of Siegel series.

Theorem 4.2 *If* $\mathrm{Re}(s) > 2n$, *we have*

$$b_\pi(T;s) = \zeta_n(k';\frac{s}{2})^{-1} \times \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} \left|N_{k'/k}(\det x_2)\right|^{\frac{s}{2}-n} |\Theta_T|\,(x).$$

We introduce the spherical function on X_T with respect to the Siegel parabolic subgroup $P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \;\middle|\; a, b, d \in M_n(k') \right\}$ by

$$\tilde{\omega}_T(x;s) = \int_K \left|N_{k'/k}(\det(kx)_2)\right|^{s-n} dk.$$

Then we have

$$\tilde{\omega}_T(x;s) = |\det T|^{s-n}\,\omega_T(x; 1-\tfrac{\pi\sqrt{-1}}{\log q}, \ldots, 1-\tfrac{\pi\sqrt{-1}}{\log q}, s-n+\tfrac{1}{2}-\tfrac{\pi\sqrt{-1}}{\log q}), \qquad (4.3)$$

which is holomorphic for $s \in \mathbb{C}$ by Theorem 1.3. Next proposition shows the relation between hermitian Siegel series and spherical functions.

Proposition 4.3 *Denote the K-orbit decomposition of $\mathfrak{X}_T(\mathcal{O}_{k'})$ as*

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \sqcup_{i=1}^r K x_i.$$

Then one has

$$b_\pi(T;s) = \zeta_n(k';\frac{s}{2})^{-1} \cdot \sum_{i=1}^r c_i \tilde{\omega}_T(x_i;\frac{s}{2}), \quad c_i = \int_{Kx_i} |\Theta_T|(y).$$

By Proposition 4.3 and Corollary 2.6, we obtain the following functional equation of hermitian Siegel series.

Theorem 4.4

$$\frac{b_\pi(T;s)}{\prod_{i=0}^{n-1}(1-(-1)^i q^{-s+i})} = \chi_\pi(\det T)^{n-1} |\det(T/2)|^{s-n} \times \frac{b_\pi(T;2n-s)}{\prod_{i=0}^{n-1}(1-(-1)^i q^{-(2n-s)+i})}.$$

References

[Bo] A. Borel: *Linear Algebraic Groups, Second enlarged edition*, Graduate Texts in Mathematics **126**, Springer, 1991.

[H1] Y. Hironaka: Spherical functions of hermitian and symmetric forms III, *Tôhoku Math. J.* **40**(1988), 651–671.

[H2] Y. Hironaka: Spherical functions and local densities on hermitian forms, *J. Math. Soc. Japan* **51**(1999), 553 – 581.

[H3] Y. Hironaka: Functional equations of spherical functions on p-adic homogeneous spaces, *Abh. Math. Sem. Univ. Hamburg* **75**(2005), 285 – 311.

[H4] Y. Hironaka: Spherical functions on p-adic homogeneous spaces, preprint, 2008, arXiv:0904.0102.

[HS] Y. Hironaka and F. Sato : The Siegel series and spherical functions on $O(2n)/(O(n)\times O(n))$, "Automorphic forms and zeta functions – Proceedings of the conference in memory of Tsuneo Arakawa –", World Scientific, 2006, p. 150 – 169.

[Ik] T. Ikeda: On the lifting of hermitian modular forms, *Comp. Math.***114** (2008), 1107-1154.

[Kr] M. L. Karel: Functional equations of Whittaker functions on p-adic groups, *Amer. J. Math.* **101**(1979), 1303 –1325.

[Om] O. T. O'Meara: *Introduction to quadratic forms*, Grund. math. Wiss. **117**, Springer-Verlag, 1973.

[KS] S. S .Kudla and W. J. Sweet: Degenerate principal series representations for $U(n,n)$, *Israel J. Math.* **98** (1997), 253 -306.

[Sch] W. Scharlau: *Quadratic and hermitian forms*, Grund. math. Wiss. **270**, Springer-Verlag, 1985.

[Sh] G. Shimura: *Euler products and Eisenstein series*, CBMS **93** (AMS), 1997.

[Ym] T. Yamazaki: Integrals defining singular series, *Memoirs Fac. Sci. Kyushu Univ.***37**(1983), 113 - 128.

Yumiko Hironaka
Department of Mathematics, Faculty of Education and Integrated Sciences,
Waseda University,
Nishi-Waseda, Tokyo 169-8050, JAPAN.